OPERATOR SCALING STABLE RANDOM FIELDS

HERMINE BIERMÉ, MARK M. MEERSCHAERT, AND HANS-PETER SCHEFFLER

ABSTRACT. A scalar valued random field $\{X(x)\}_{x \in \mathbb{R}^d}$ is called *operator-scaling* if for some $d \times d$ matrix E with positive real parts of the eigenvalues and some $H > 0$ we have
$$\{X(c^E x)\}_{x \in \mathbb{R}^d} \overset{f.d.}{=} \{c^H X(x)\}_{x \in \mathbb{R}^d} \quad \text{for all } c > 0,$$
where $\overset{f.d.}{=}$ denotes equality of all finite-dimensional marginal distributions. We present a moving average and a harmonizable representation of stable operator scaling random fields by utilizing so called E-homogeneous functions φ, satisfying $\varphi(c^E x) = c\varphi(x)$. These fields also have stationary increments and are stochastically continuous. In the Gaussian case critical Hölder-exponents and the Hausdorff-dimension of the sample paths are also obtained.

1. INTRODUCTION

A scalar valued random field $\{X(x)\}_{x \in \mathbb{R}^d}$ is called *operator-scaling* if for some $d \times d$ matrix E with positive real parts of the eigenvalues and some $H > 0$ we have

$$(1.1) \qquad \{X(c^E x)\}_{x \in \mathbb{R}^d} \overset{f.d.}{=} \{c^H X(x)\}_{x \in \mathbb{R}^d} \quad \text{for all } c > 0,$$

where $\overset{f.d.}{=}$ denotes equality of all finite-dimensional marginal distributions. As usual $c^E = \exp(E \log c)$ where $\exp(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!}$ is the matrix exponential. Note that if $E = I$, the identity matrix, then (1.1) is just the well-known *self-similarity* property $\{X(cx)\}_{x \in \mathbb{R}^d} \overset{f.d.}{=} \{c^H X(x)\}_{x \in \mathbb{R}^d}$ where one usually calls H the Hurst-index. See [11] for an overview of self-similar processes in the one-dimensional case $d = 1$. Self-similar processes are used in various fields of applications such as internet traffic modelling [23], ground water modelling and mathematical finance, just to mention a few. Various examples can be found for instance in the books [18] and [1]. A very

important class of such fields or processes are the *fractional stable fields* and especially the Lévy fractional Brownian field.

These fields have different definitions which are usually not equivalent. More precisely, for $0 < \alpha \le 2$ let $Z_\alpha(dy)$ be an independently scattered symmetric α-stable $(S\alpha S)$ random measure on $\mathbb{R}^d$ with Lebesgue control measure λ^d (see [21] p. 121). For $0 < H < 1$ one defines the *moving average* representation by

$$(1.2) \qquad X_H(x) = \int_{\mathbb{R}^d} \left(\|x - y\|^{H-d/\alpha} - \|y\|^{H-d/\alpha} \right) Z_\alpha(dy).$$

For $W_\alpha(d\xi)$ a complex isotropic $S\alpha S$ random measure with Lebesgue control measure the harmonizable representation is given by

$$(1.3) \qquad \tilde{X}_H(x) = \mathrm{Re} \int_{\mathbb{R}^d} \left(e^{i\langle x,\xi \rangle} - 1 \right) \|\xi\|^{-H-d/\alpha} W_\alpha(d\xi).$$

See [21] for a comprehensive introduction to random integrals with respect to stable measures. It follows from basic properties that $\{X_H(cx)\}_{x\in\mathbb{R}^d} \overset{f.d.}{=} \{c^H X(x)\}_{x\in\mathbb{R}^d}$ as well as $\{\tilde{X}_H(cx)\}_{x\in\mathbb{R}^d} \overset{f.d.}{=} \{c^H \tilde{X}_H(x)\}_{x\in\mathbb{R}^d}$. Moreover, both processes have *stationary increments*, that is for any $h \in \mathbb{R}^d$ we have $\{X_H(x+h)-X_H(h)\}_{x\in\mathbb{R}^d} \overset{f.d.}{=} \{X_H(x)\}_{x\in\mathbb{R}^d}$ and similarly for $\{\tilde{X}_H(x)\}_{x\in\mathbb{R}^d}$. Furthermore both fields are *isotropic*, that is $\{X_H(Ax)\}_{x\in\mathbb{R}^d} \overset{f.d.}{=} \{X(x)\}_{x\in\mathbb{R}^d}$ for any orthogonal matrix A. It is worth mentioning that if $\alpha < 2$ the fields $\{X_H(x)\}_{x\in\mathbb{R}^d}$ and $\{\tilde{X}_H(x)\}_{x\in\mathbb{R}^d}$ defined in (1.2) and (1.3), respectively, are usually different. See [21], Theorem 7.7.4 for the one-dimensional case. However, in the Gaussian case $\alpha = 2$, by computing the covariance function of the fields, it follows that $\{X_H(x)\}_{x\in\mathbb{R}^d}$ and $\{\tilde{X}_H(x)\}_{x\in\mathbb{R}^d}$ have the same law up to a multiplicative constant and known as the Lévy fractional Brownian field.

Certain applications (see, e.g., [7, 8, 20] and references therein) require that the random field is anisotropic and satisfies a scaling relation. This scaling relation should have different Hurst indices in different directions and these directions should not necessarily be orthogonal. In the Gaussian case a prominent example of an anisotropic random field is the fractional Brownian sheet $\{B_H(x)\}_{x\in\mathbb{R}^d}$ defined as follows: Let $0 < H_j < 1$ for $j = 1, \ldots, d$ and set $H = (H_1, \ldots, H_d)$. Define

$$B_H(x) = \int_{\mathbb{R}^d} \prod_{j=1}^{d} \left[|x_j - u_j|^{H_j - 1/2} - |u_j|^{H_j - 1/2} \right] Z_2(du).$$

See [5, 14, 24] and the literature cited there for more information on these fields. Then, if we set $E = \text{diag}(H_1^{-1}, \ldots, H_d^{-1})$, it follows by a simple computation that $\{B_H(c^E x)\}_{x \in \mathbb{R}^d} \overset{f.d.}{=} \{cB_H(x)\}_{x \in \mathbb{R}^d}$ so $\{B_H(x)\}_{x \in \mathbb{R}^d}$ is operator scaling in the sense of (1.1). However, $\{B_H(x)\}_{x \in \mathbb{R}^d}$ does not have stationary increments.

The purpose of this paper is to define two different classes of *operator scaling stable random fields* (OSSRF) and analyze their basic properties. We present a moving average representation as well as a harmonizable representation. Our constructions are based on a class of E-homogeneous functions $\varphi : \mathbb{R}^d \to [0, \infty)$ where φ is positive on $\mathbb{R}^d \setminus \{0\}$ and $\varphi(c^E x) = c\varphi(x)$ for all $x \in \mathbb{R}^d$ and $c > 0$. Such functions were studied in detail in [19], Chapter 5. It will turn out that the harmonizable representation allows more flexibility in the class of possible functions φ in contrast to the moving average representation which is more restrictive. However, in both cases the OSSRFs satisfy (1.1), have stationary increments and are continuous in probability. In the Gaussian case $\alpha = 2$ we show that there exists modifications of these fields which are almost surely Hölder-continuous of certain indices and we compute the Hausdorff-dimension of the graph.

This paper is organized as follows: In section 2 we introduce the class of E-homogeneous functions, derive some basic properties and provide important examples. In section 3 we define and analyze a moving average representation of OSSRFs. Section 4 is devoted to the harmonizable representation and its properties. Finally, in the Gaussian case $\alpha = 2$, we analyze the sample path properties of both the moving average and the harmonizable representation of OSSRFs.

2. E-HOMOGENEOUS FUNCTIONS

Let E be a real $d \times d$ matrix with positive real parts of the eigenvalues $0 < a_1 < a_2 \ldots < a_p$ for $p \leq d$. Let us denote $\Gamma = \mathbb{R}^d \setminus \{0\}$. It follows from Lemma 6.1.5 of [19] that there exists a norm $\|\cdot\|_0$ on $\mathbb{R}^d$ such that for the unit sphere $S_0 = \{x \in \mathbb{R}^d : \|x\|_0 = 1\}$ the mapping $\Psi : (0, \infty) \times S_0 \to \Gamma$, $\Psi(r, \theta) = r^E \theta$ is a homeomorphism. Moreover for any $x \in \Gamma$ the function $t \mapsto \|t^E x\|_0$ is strictly increasing. Hence we can write any $x \in \Gamma$ uniquely as $x = \tau(x)^E l(x)$ for some *radial part* $\tau(x) > 0$ and some direction $l(x) \in S_0$ such that $x \mapsto \tau(x)$ and $x \mapsto l(x)$ are continuous. Observe that $S_0 = \{x \in \mathbb{R}^d : \tau(x) = 1\}$ is compact. Moreover we know that $\tau(x) \to \infty$ as

$x \to \infty$ and $\tau(x) \to 0$ as $x \to 0$. Hence we can extend $\tau(\cdot)$ continuously by setting $\tau(0) = 0$. Note that further $\tau(-x) = \tau(x)$ and $l(-x) = -l(x)$. The following result gives bounds on the growth rate of $\tau(x)$ in terms of the real parts of the eigenvalues of E.

Lemma 2.1. *For any (small) $\delta > 0$ there exist constants $C_1, \ldots, C_4 > 0$ such that for all $\|x\|_0 \le 1$ or all $\tau(x) \le 1$,*

$$C_1 \|x\|_0^{1/a_1 + \delta} \le \tau(x) \le C_2 \|x\|_0^{1/a_p - \delta},$$

and, for all $\|x\|_0 \ge 1$ or all $\tau(x) \ge 1$,

$$C_3 \|x\|_0^{1/a_p - \delta} \le \tau(x) \le C_4 \|x\|_0^{1/a_1 + \delta}.$$

Proof. We will only prove the first two inequalities. It follows from Theorem 2.2.4 of [19] that for any $\delta' > 0$ we have $t^{a_1 - \delta'} \|t^{-E}\theta\|_0 \to 0$ as $t \to \infty$ uniformly in $\|\theta\|_0 = 1$. Hence $\|t^{-E}\|_0 := \sup_{\theta \in S_0} \|t^{-E}\theta\|_0 \le C t^{-a_1 + \delta'}$ for all $t \ge 1$ and some constant $C > 0$. Equivalently $\|s^E\|_0 \le C s^{a_1 - \delta'}$ for all $s \le 1$. Since $\|x\|_0 = \|\tau(x)^E l(x)\|_0 \le \|\tau(x)^E\|_0 \le C\tau(x)^{a_1 - \delta'}$ we get $\tau(x) \ge C_1 \|x\|_0^{1/a_1 + \delta}$, for $\delta = \frac{1}{a_1 - \delta_1} - \frac{1}{a_1}$, if $\|x\|_0 \le 1$ which is equivalent to $\tau(x) \le 1$.

Similarly we know that, for any $\delta' > 0$, $t^{-a_p - \delta'} \|t^E \theta\|_0 \to 0$ as $t \to \infty$ uniformly in $\|\theta\|_0 = 1$. Therefore $\|t^E\|_0 \le C t^{a_p + \delta'}$ for all $t \ge 1$ or equivalently $\|s^{-E}\|_0 \le C s^{-a_p - \delta'}$ for all $s \le 1$. But $x = \tau(x)^E l(x)$ and $l(x) = \tau(x)^{-E} x$. Thus, $1 \le \|\tau(x)^{-E}\|_0 \|x\|_0$ and $\|x\|_0 \ge C^{-1} \tau(x)^{a_p + \delta'}$ for all $\|x\|_0 \le 1$. Hence $\tau(x) \le C_2 \|x\|_0^{1/a_p - \delta}$ for $\delta = \frac{1}{a_p} - \frac{1}{a_p + \delta'}$ and $\|x\|_0 \le 1$. The proof is complete. $\qquad\square$

The following results generalize some of the results in [13], Chapter 1.A to our more general case of exponents E.

Lemma 2.2. *There exists a constant $K \ge 1$ such that for all $x, y \in \mathbb{R}^d$ we have*

$$\tau(x + y) \le K\big(\tau(x) + \tau(y)\big).$$

Proof. Observe that the set $G = \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : \tau(x) + \tau(y) = 1\}$ is bounded by Lemma 2.1 and closed by continuity of τ. Hence G is a compact set. Thus the continuous function $(x, y) \mapsto \tau(x + y)$ assumes a finite maximum K on G. Since $S_0 \times \{0\} \subset G$, we have $K \ge 1$. Given any $x, y \in \mathbb{R}^d$ both not equal to zero we set

$s = (\tau(x) + \tau(y))^{-1}$. Then, with $\tau(c^E x) = c\tau(x)$ it follows that

$$\tau(x + y) = s^{-1}\tau\left(s^E(x + y)\right) = s^{-1}\tau\left((s^E x) + (s^E y)\right).$$

But $\left(s^E x, s^E y\right) \in G$ since $\tau\left(s^E x\right) + \tau\left(s^E y\right) = s\left(\tau(x) + \tau(y)\right) = 1$. Therefore,

$$\tau(x + y) \le Ks^{-1} = K\left(\tau(x) + \tau(y)\right)$$

and the proof is complete. $\qquad\square$

Now let $q = \text{trace}(E)$ and observe that by multivariable change of variables we have $\lambda^d(c^E(B)) = c^q\lambda^d(B)$ for all Borel sets $B \subset \mathbb{R}^d$, $c > 0$, which can be written as $d(c^E x) = c^q dx$. Let $B(r, x) = \{y \in \mathbb{R}^d : \tau(y - x) < r\}$ denote the *ball* of radius $r > 0$ around $x \in \mathbb{R}^d$. Then it is easy to see that $B(r, x) = x + B(r, 0) = x + r^E B(1, 0)$ and hence $\lambda^d(B(r, x)) = r^q\lambda^d(B(1, 0))$. The following proposition provides an integration in *polar coordinates* formula.

Proposition 2.3. *There exists a unique finite Radon measure σ on S_0 such that for all $f \in L^1(\mathbb{R}^d, dx)$ we have*

$$\int_{\mathbb{R}^d} f(x)\,dx = \int_0^\infty \int_{S_0} f(r^E \theta)\,\sigma(d\theta)\,r^{q-1}\,dr.$$

The proof of Proposition 2.3 is based on the following.

Lemma 2.4. *If $f : \Gamma \to \mathbb{C}$ is continuous and $f(r^E x) = r^{-q}f(x)$ for all $r > 0$ and $x \in \Gamma$, then there exists a constant μ_f such that for all $g \in L^1\left((0, \infty), r^{-1}dr\right)$ we have*

$$\int_{\mathbb{R}^d} f(x)g(\tau(x))\,dx = \mu_f \int_0^\infty g(r)\frac{dr}{r}.$$

Proof. Let $L_f : (0, \infty) \to \mathbb{C}$ be defined as

$$L_f(r) = \begin{cases} \int_{1 \le \tau(x) \le r} f(x)\,dx & \text{if } r \ge 1 \\ -\int_{1 \le \tau(x) \le r^{-1}} f(x)\,dx & \text{if } r < 1\,. \end{cases}$$

Since f is continuous on Γ, from dominated convergence, L_f is continuous on $(0, 1) \cup (1, +\infty)$. But $\lambda^d(B(r, 0)) = r^q\lambda^d(B(1, 0))$ implies that $\lambda^d\left(\{x \in \mathbb{R}^d : \tau(x) = r\}\right) = 0$, and it follows that L_f is also continuous at point 1 and thus on $(0, +\infty)$. Moreover, for any $r > 0$ we have $L_f(r^{-1}) = -L_f(r)$. When $rs \ge 1$ with $r, s > 0$ a change of variables yields

$$L_f(rs) = \int_{1 \le \tau(x) \le rs} f(x)\,dx = \int_{1 \le \tau(s^E y) \le rs} f(s^E y)s^q\,dy = \int_{s^{-1} \le \tau(y) \le r} f(y)\,dy.$$

Let us assume for instance that $1 \leq s^{-1} \leq r$. Then, by continuity of L_f,

$$\int_{s^{-1} \leq \tau(y) \leq r} f(y)\, dy = \int_{1 \leq \tau(y) \leq r} f(y)\, dy - \int_{1 \leq \tau(y) \leq s^{-1}} f(y)\, dy = L_f(r) - L_f(s^{-1}).$$

It follows using $L_f(s^{-1}) = -L_f(s)$ that

(2.1) $$L_f(rs) = L_f(r) + L_f(s).$$

Similarly we show that (2.1) holds for $s^{-1} \leq 1 \leq r$ and $s^{-1} \leq r \leq 1$ and thus for all $rs \geq 1$. Using again the fact that $L_f(r^{-1}) = -L_f(r)$, for all $r > 0$, (2.1) is valid for all $r, s > 0$. By continuity of L_f it follows that $L_f(r) = L_f(e) \log r$. We set $\mu_f = L_f(e)$. If $g(r) = 1_{]a,b]}(r)$ for some $0 < a < b$ we get

$$\begin{aligned}
\int_{\mathbb{R}^d} f(x)g(\tau(x))\, dx &= \int_{a < \tau(x) \leq b} f(x)\, dx = L_f(b) - L_f(a) \\
&= \mu_f\big(\log b - \log a\big) = \mu_f \int_0^\infty g(r)\, \frac{dr}{r}.
\end{aligned}$$

The general result follows by taking linear combinations and limits of these functions in the standard way. $\square$

Proof of Proposition 2.3. When $f \in C(S_0)$ define $\tilde{f}$ on Γ by $\tilde{f}(x) = \tau(x)^{-q} f\big(l(x)\big)$. The function $\tilde{f}$ satisfies the hypothesis of Lemma 2.4. If $f \geq 0$ then $\mu_{\tilde{f}} = L_{\tilde{f}}(e) = \int_{1 \leq \tau(x) \leq e} \tau(x)^{-q} f\big(l(x)\big)\, dx \geq 0$. Moreover $\mu_{a\tilde{f}} = a\mu_{\tilde{f}}$, $\mu_{\tilde{f}+\tilde{g}} = \mu_{\tilde{f}} + \mu_{\tilde{g}}$ and the mapping $f \mapsto \mu_{\tilde{f}}$ is continuous. Hence this mapping is a positive linear functional on $C(S_0)$. Therefore there exists a Radon measure σ on S_0 such that $\mu_{\tilde{f}} = \int_{S_0} f(\theta)\, \sigma(d\theta)$.

If $g_1 \in C_c((0, \infty))$ we get from applying Lemma 2.4 with $\tilde{f}$ and $g(r) = r^q g_1(r)$ that

$$\begin{aligned}
\int_{\mathbb{R}^d} f(l(x))g_1(\tau(x))\, dx &= \int_{\mathbb{R}^d} \tilde{f}(x)\tau(x)^q g_1(\tau(x))\, dx \\
&= \mu_f \int_0^\infty g_1(r) r^{q-1}\, dr \\
&= \int_0^\infty \int_{S_0} f(\theta)\, \sigma(d\theta)\, g_1(r) r^{q-1}\, dr.
\end{aligned}$$

Since linear combinations of functions of the form $f(l(x))g_1(\tau(x))$ are dense in $L^1(\mathbb{R}^d, dx)$ the result follows. $\square$

Corollary 2.5. *Let $\beta \in \mathbb{R}$ and suppose $f : \mathbb{R}^d \to \mathbb{C}$ is measurable such that $|f(x)| = O(\tau(x)^\beta)$. If $\beta > -q$ then f is integrable near 0, and if $\beta < -q$ then f is integrable near infinity.*

We are now in position to define the class of E-homogeneous functions and a important subclass needed in the moving average representation of OSSRFs. Let E be a $d \times d$ matrix as above such that $0 < a_1 < \cdots < a_p$ and for $x \in \Gamma$ let $\big(\tau(x), l(x)\big)$ be the *polar coordinates* associated to E, that is $x = \tau(x)^E l(x)$.

Definition 2.6. Let $\varphi : \mathbb{R}^d \to \mathbb{C}$ be any function. We say that φ is E-*homogeneous* if $\varphi(c^E x) = c\varphi(x)$ for all $c > 0$ and $x \in \Gamma$.

It follows that an E-homogeneous function φ is completely determined by its values on S_0, since $\varphi(x) = \varphi(\tau(x)^E l(x)) = \tau(x)\varphi(l(x))$. Observe that if φ is E-homogeneous and continuous with positive values on Γ, then

$$(2.2) \qquad M_\varphi = \max_{\theta \in S_0} \varphi(\theta) > 0 \text{ and } m_\varphi = \min_{\theta \in S_0} \varphi(\theta) > 0.$$

Moreover by continuity we necessarily have $\varphi(0) = 0$.

Definition 2.7. Let $\beta > 0$. A continuous function $\varphi : \mathbb{R}^d \to [0, \infty)$ is called (β, E)-*admissible*, if $\varphi(x) > 0$ for all $x \neq 0$ and for any $0 < A < B$ there exists a positive constant $C > 0$ such that, for $A \leq \|y\| \leq B$,

$$\tau(x) \leq 1 \Rightarrow |\varphi(x + y) - \varphi(y)| \leq C\tau(x)^\beta.$$

Remark 2.8. If a continuous function $\varphi : \mathbb{R}^d \to [0, \infty)$ is positive and Lipschitz on Γ, that is $|\varphi(x) - \varphi(y)| \leq C\|x - y\|_0$ for $x, y \in \Gamma$, then φ is (β, E)-admissible for all $\beta < a_1$ by Lemma 2.1.

Remark 2.9. If φ is (β, E)-admissible then $\beta \leq a_1$. In fact, if $\mathbb{R}^d = V_1 \oplus \cdots \oplus V_p$ is the spectral decomposition of $\mathbb{R}^d$ with respect to E (see [19], Chapter 2 for details), by restricting the argument of the proof of Lemma 2.1 to the space V_1 on can show that for any $\delta > 0$ there exists a constant $C > 0$ such that $\tau(x) \leq C\|x\|_0^{1/a_1 - \delta}$ for all $x \in V_1$ with $\|x\|_0 \leq 1$. Then, if for some fixed nonzero $u \in V_1$ we consider the function $t \mapsto \varphi(tu)$ we get for $\delta_1 = \beta\delta$ that $|\varphi(tu + su) - \varphi(su)| \leq C|t|^{\beta/a_1 - \delta_1}$ for all small t and s bounded away from zero and infinity. If one would have $\beta > a_1$, one could chose $\delta > 0$ such that $\beta/a_1 - \delta_1 > 1$ and hence there would exist a constant $K > 0$ such that $\varphi(tu) = K$ for all $t \neq 0$. But since φ is continuous and $\varphi(0) = 0$ this is impossible.

Remark 2.10. In general the exponent E of a homogeneous function φ is not unique. It is easy to check that $\varphi(x) \to \infty$ as $\|x\| \to \infty$, and then Theorem 5.2.13 in [19] implies that the set of possible exponents is $E + T\mathcal{S}(\varphi)$ where E is any exponent, $\mathcal{S}(\varphi)$ is the set of symmetries of φ, and $T\mathcal{S}(\varphi)$ is the tangent space at the identity. Here we say that A is a symmetry of φ if $\varphi(Ax) = \varphi(x)$ for all $x \in \mathbb{R}^d$. The symmetries $\mathcal{S}(\varphi)$ form a Lie group, and the tangent space consists of all derivatives $x'(0)$ of smooth curves $x(t)$ on $\mathcal{S}(\varphi)$ for which $x(0) = I$ the identity. For example, if φ is rotationally invariant then $\mathcal{S}(\varphi)$ is the orthogonal group and $T\mathcal{S}(\varphi)$ is the linear space of skew-symmetric matrices. Although exponents are not unique, Theorem 5.2.14 in [19] shows that every exponent E of a homogeneous function φ has the same real spectrum $0 < a_1 < \cdots < a_p$ and induces the same spectral decomposition $\mathbb{R}^d = V_1 \oplus \ldots \oplus V_p$, since these structural components describe the growth properties of the homogeneous function. In particular, the function $r \mapsto \varphi(rx)$ grows like r^{1/a_i} for any nonzero $x \in V_i$, see Section 5.3 in [19] for more details.

We conclude this section by examples of (β, E)-admissible, E-homogeneous functions $\varphi : \mathbb{R}^d \to [0, \infty)$ used in Theorem 3.1 below to define a moving average representation of OSSRFs $\{X_\varphi(x)\}_{x \in \mathbb{R}^d}$. Let us denote $< ., . >$ the standard inner product on $\mathbb{R}^d$ and E^t the transpose of any $d \times d$-matrix E with respect to this inner product. The following class of examples is inspired by the log-characteristic function of a full operator stable law on $\mathbb{R}^d$. See [19] for details.

Theorem 2.11. *Assume E is a real $d \times d$-matrix such that the real parts of the eigenvalues satisfy $1/2 < a_1 < a_2 \ldots < a_p$ for $p \leq d$. Assume $M(d\theta)$ is a finite measure on the unit sphere S_0 corresponding to E such that*

$$\text{span}\{r^{E^t}\theta : r > 0, \theta \in \text{supp}(M)\} = \mathbb{R}^d.$$

Then

$$\varphi(x) = \int_{S_0} \int_0^\infty \left(1 - \cos(\langle x, r^{E^t}\theta\rangle)\right) \frac{dr}{r^2} \, M(d\theta)$$

is a continuous, E-homogeneous function such that $\varphi(x) > 0$ for all $x \in \Gamma$. Moreover φ is (β, E)-admissible for $\beta < \min\left(a_1, \frac{a_1}{a_p}\right)$ if $a_1 \leq 1$ and $\beta = 1$ if $a_1 > 1$.

Proof. Let $a_1 > 1/2$ denote the smallest real part of the eigenvalues of E. Since E and E^t have the same eigenvalues, it follows from Theorem 2.2.4 of [19] that

for any $\delta > 0$ there exists a constant $C > 0$ such that $\|r^{E^t}\theta\|_0 \leq Cr^{a_1-\delta}$ for all $0 < r \leq 1$ and $\theta_0 \in S_0$. Therefore, from dominated convergence, φ is well-defined and continuous on $\mathbb{R}^d$. Moreover we have $\varphi(x) \geq 0$ and $\varphi(x) = 0$ implies $x = 0$. A simple change of variable shows that $\varphi(c^E x) = c\varphi(x)$ for all $c > 0$ and $x \in \mathbb{R}^d$. It remains to show that φ is (β, E)-admissible. Using the trigonometric identity $\cos(a) - \cos(b) = -2\sin((a+b)/2)\sin((a-b)/2)$ we have for any $x, y \in \mathbb{R}^d$ that

$$(2.3) \quad |\varphi(x+y) - \varphi(y)| \leq 2 \int_{S_0} \int_0^\infty \left| \sin\left(\frac{\langle x + 2y, r^{E^t}\theta \rangle}{2}\right) \sin\left(\frac{\langle x, r^{E^t}\theta \rangle}{2}\right) \right| \frac{dr}{r^2} M(d\theta).$$

First, let us assume that $a_1 > 1$, then an upper bound of (2.3) is given by

$$2 \int_{S_0} \int_0^\infty \left| \sin\left(\frac{\langle x, r^{E^t}\theta \rangle}{2}\right) \right| \frac{dr}{r^2} M(d\theta),$$

which is finite because $a_1 > 1$, using $\|r^{E^t}\theta\|_0 \leq Cr^{a_1-\delta}$ for all $0 < r \leq 1$ and $\theta_0 \in S_0$, and elementary estimates. Moreover writing $x = \tau(x)^E l(x)$ a change of variables yields to

$$2 \int_{S_0} \int_0^\infty \left| \sin\left(\frac{\langle x, r^{E^t}\theta \rangle}{2}\right) \right| \frac{dr}{r^2} M(d\theta) = 2\tau(x) \int_{S_0} \int_0^\infty \left| \sin\left(\frac{\langle l(x), r^{E^t}\theta \rangle}{2}\right) \right| \frac{dr}{r^2} M(d\theta),$$

which proves that φ is 1-admissible.

Let us now consider the case where $a_1 \leq 1$. Choose $\delta > 0$ small enough. On one hand, for $r \leq 1$, one can find $C > 0$ such that

$$\left| \sin\left(\frac{\langle x + 2y, r^{E^t}\theta \rangle}{2}\right) \sin\left(\frac{\langle x, r^{E^t}\theta \rangle}{2}\right) \right| \leq C \left(2\|y\|_0 + \|x\|_0 \right) \|x\|_0 r^{2a_1-\delta}.$$

On the other hand, it follows from Theorem 2.2.4 of [19] that one can find $C > 0$ such that $\|r^{E^t}\theta\|_0 \leq Cr^{a_p+\delta}$ for all $r \geq 1$ and $\theta_0 \in S_0$. Thus, for $\gamma < \min\left(1, \frac{1}{a_p}\right)$, using $|\sin(u)| \leq |u|^\gamma$, one can find $C > 0$ such that

$$\left| \sin\left(\frac{\langle x + 2y, r^{E^t}\theta \rangle}{2}\right) \sin\left(\frac{\langle x, r^{E^t}\theta \rangle}{2}\right) \right| \leq C\|x\|_0^\gamma r^{\gamma a_p + \gamma\delta}.$$

Therefore, by substituting these upper bounds into the right-hand side of (2.3) and integrating, for some constant $C > 0$ we have shown that $|\varphi(x+y) - \varphi(y)| \leq C\|x\|_0^\gamma$ for all $\|x\|_0 \leq 1$ and $A \leq \|y\|_0 \leq B$.

Since by Lemma 2.1 $\|x\|_0 \leq C\tau(x)^{a_1-\delta}$ for $\tau(x) \leq 1$, the assertion follows with $\beta = \gamma(a_1 - \delta)$. $\qquad \square$

The following result gives a constructive description of a large class of continuous, admissible E-homogeneous functions.

Corollary 2.12. *Let $\theta_1, \ldots, \theta_d$ be any basis of $\mathbb{R}^d$, let $0 < \lambda_1 \leq \cdots \leq \lambda_d$ and $C_1, \ldots, C_d > 0$. Choose a $d \times d$ matrix E such that $E^t \theta_j = \lambda_j \theta_j$ for $j = 1, \ldots, d$. Then for any $\rho > 0$, if $\rho < 2\lambda_1$ the function*

$$\varphi(x) = \Big(\sum_{j=1}^{d} C_j |\langle x, \theta_j \rangle|^{\rho/\lambda_j} \Big)^{1/\rho}$$

is a continuous E-homogeneous and (β, E)-admissible function for $\beta < \min\left(\lambda_1, \rho\frac{\lambda_1}{\lambda_d}\right)$ if $\lambda_1 \leq \rho$ and $\beta = \rho$ if $\lambda_1 > \rho$.

Proof. First observe that since $r^{E^t}\theta_j = r^{\lambda_j}\theta_j$ it follows that $\varphi(c^E x) = c\varphi(x)$. Moreover φ is continuous. Let $B > A > 0$, since $y \mapsto \sum_{j=1}^{d} C_j |\langle y, \theta_j \rangle|^{\rho/\lambda_j}$ is continuous and positive on Γ, by the mean value theorem, for $A \leq \|y\| \leq B$ and $\|x\| \leq A/2$, one can find $C > 0$ such that

$$(2.4) \qquad |\varphi(x + y) - \varphi(y)| \leq C \Big| \sum_{j=1}^{d} C_j |\langle x + y, \theta_j \rangle|^{\rho/\lambda_j} - \sum_{j=1}^{d} C_j |\langle y, \theta_j \rangle|^{\rho/\lambda_j} \Big|.$$

Hence it remains to show that the right hand side of (2.4) is (β, E)-admissible. Let $M = \sum_{j=1}^{d} \gamma_j \varepsilon_{\theta_j}$ for suitable $\gamma_j > 0$, where ε_θ denotes the dirac mass in θ. Let us define for $x \in \mathbb{R}^d$,

$$\psi(x) = \int_{S_0} \int_0^\infty \big(1 - \cos(\langle x, r^{(1/\rho)E^t}\theta \rangle)\big) \frac{dr}{r^2} M(d\theta),$$

which is well defined since $\rho < 2\lambda_1$. Moreover, by Theorem 2.11, ψ is $(\beta, (1/\rho)E)$-admissible for $\beta < \min\left(\frac{\lambda_1}{\rho}, \frac{\lambda_1}{\lambda_d}\right)$ if $\lambda_1 < \rho$ and $\beta = 1$ if $\lambda_1 > \rho$. Let $\tau_\rho(x)$ denote the radial part with respect to $(1/\rho)E$. Then uniqueness implies that the radial part $\tau(x)$ with respect to E is given by $\tau(x) = \tau_\rho(x)^{1/\rho}$. Hence ψ is (β, E)-admissible for $\beta < \min\left(\lambda_1, \rho\frac{\lambda_1}{\lambda_d}\right)$ if $\lambda_1 < \rho$ and $\beta = \rho$ if $\lambda_1 > \rho$.

Moreover, since $r^{(1/\rho)E^t}\theta_j = r^{\lambda_j/\rho}\theta_j$ we get

$$\psi(x) = \sum_{j=1}^d \gamma_j \int_0^\infty \big(1 - \cos\big(r^{\lambda_j/\rho}|\langle x, \theta_j\rangle|\big)\big) \frac{dr}{r^2}$$

$$= \sum_{j=1}^d \frac{\rho\gamma_j}{\lambda_j} \Big(\int_0^\infty \big(1 - \cos(s)\big) s^{-(\rho/\lambda_j)-1}\, ds\Big)|\langle x, \theta_j\rangle|^{\rho/\lambda_j}$$

$$= \sum_{j=1}^d C_j|\langle x, \theta_j\rangle|^{\rho/\lambda_j}.$$

This completes the proof. $\hfill\square$

3. Moving average representation

In this section we consider a moving average representation of OSSRFs and derive its basic properties. We first give sufficient conditions such that the integral representation exists. More precisely, for $0 < \alpha \le 2$ we consider $Z_\alpha(dy)$ an independently scattered $S\alpha S$ random measure on $\mathbb{R}^d$ with Lebesgue control measure λ^d. Then we define a moving average representation of OSSRFs using the basic fact that a random integral $\int_{\mathbb{R}^d} f(y)\, Z_\alpha(dy)$ exists if and only if $\int_{\mathbb{R}^d} |f(y)|^\alpha\, dy < \infty$.

Throughout this section we fix a real $d \times d$ matrix E with $0 < a_1 < \cdots < a_p$ denoting the real parts of the eigenvalues of E. As before, let $q = \text{trace}(E)$.

Theorem 3.1. *Let $\beta > 0$. Let $\varphi : \mathbb{R}^d \to [0,\infty)$ be an E-homogeneous, (β, E)-admissible function. Then for any $0 < \alpha \le 2$ and any $0 < H < \beta$ the random field*

$$(3.1) \qquad X_\varphi(x) = \int_{\mathbb{R}^d} \Big(\varphi(x - y)^{H-q/\alpha} - \varphi(-y)^{H-q/\alpha}\Big) Z_\alpha(dy)\,, x \in \mathbb{R}^d$$

exists and is stochastically continuous.

Proof. Let us recall that $X_\varphi(x)$ exists if and only if

$$\Gamma_\varphi^\alpha(x) = \int_{\mathbb{R}^d} \Big|\varphi(x - y)^{H-q/\alpha} - \varphi(-y)^{H-q/\alpha}\Big|^\alpha dy < \infty.$$

Let us assume that $H \in (0, \beta)$. Observe that by (2.2) and the fact that φ is E-homogeneous, $\varphi(z) \le M_\varphi \tau(z)$ and $\varphi(z) \ge m_\varphi \tau(z)$ for all $z \ne 0$. Fix any $x \in \Gamma$. Then,

$$\Big|\varphi(x - y)^{H-q/\alpha} - \varphi(-y)^{H-q/\alpha}\Big|^\alpha \le C\Big(\tau(x-y)^{\alpha H-q} + \tau(y)^{\alpha H-q}\Big).$$

But for any $R > 0$ it follows from Corollary 2.5 that $\int_{\tau(y) \leq R} \tau(y)^{\alpha H - q} \, dy < \infty$ if $H > 0$. Moreover, by Lemma 2.2 $\{y : \tau(x - y) \leq R\} \subset \{y : \tau(y) \leq K(R + \tau(x))\}$ and hence, by a change of variable we obtain using Corollary 2.5 again that, if $H > 0$

$$\int_{\tau(y) \leq R} \tau(x - y)^{\alpha H - q} \, dy = \int_{\tau(x-y) \leq R} \tau(y)^{\alpha H - q} \, dy \leq \int_{\tau(y) \leq K(R + \tau(x))} \tau(y)^{\alpha H - q} \, dy < \infty.$$

It remains to show that for some $R = R(x) > 0$ we have

$$(3.2) \qquad \int_{\tau(y) > R} \left| \varphi(x + y)^{H - q/\alpha} - \varphi(y)^{H - q/\alpha} \right|^{\alpha} \, dy < \infty.$$

Observe that for $\tau(y) > R$, $\varphi(y) > 0$, so we can write

$$\varphi(x + y) = \varphi \left(\varphi(y)^E \left(\varphi(y)^{-E} x + \varphi(y)^{-E} y \right) \right) = \varphi(y) \varphi \left(\varphi(y)^{-E} x + \varphi(y)^{-E} y \right),$$

since φ is E-homogeneous. Moreover $\varphi \left(\varphi(y)^{-E} y \right) = 1$ and since φ is (β, E) admissible, one can find $C > 0$ such that

$$\left| \varphi \left(\varphi(y)^{-E} x + \varphi(y)^{-E} y \right) - 1 \right| \leq C \tau \left(\varphi(y)^{-E} x \right)^{\beta} = C \varphi(y)^{-\beta} \tau(x)^{\beta}.$$

Hence by the mean value theorem applied to the function $t^{H - q/\alpha}$ near $t = 1$, one can find $C_1 > 0$ such that

$$\left| \varphi(x + y)^{H - q/\alpha} - \varphi(y)^{H - q/\alpha} \right| = \varphi(y)^{H - q/\alpha} \left| \varphi(\varphi(y)^{-E} x + \varphi(y)^{-E} y)^{H - q/\alpha} - 1 \right|$$
$$\leq C_1 \varphi(y)^{H - \beta - q/\alpha} \tau(x)^{\beta},$$

for all $\tau(y) > R$, where $R > 0$ is chosen sufficiently large so that $C \varphi(y)^{-\beta} \tau(x)^{\beta} < 1/2$ for all $\tau(y) > R$. But $\varphi(y)^{H - \beta - q/\alpha} \leq C_2 \tau(y)^{H - \beta - q/\alpha}$ and by Corollary 2.5 we know that $\int_{\tau(y) \geq R} \tau(y)^{\alpha H - q - \alpha \beta} \, dy < \infty$ if $H < \beta$. This allows to conclude that $\Gamma_{\varphi}^{\alpha}(x)$ is finite for all $x \in \mathbb{R}^d$. Let us now show that X_{φ} is stochastically continuous. Since X_{φ} is a $S\alpha S$ field, it follows from Proposition 3.5.1 in [21] that X_{φ} is stochastically continuous if and only if, for all $x_0 \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} \left| \varphi(x_0 + x - y)^{H - q/\alpha} - \varphi(x_0 - y)^{H - q/\alpha} \right|^{\alpha} \, dy \to 0 \quad \text{as } x \to 0.$$

By a change a variables, this holds if and only if

$$(3.3) \qquad \Gamma_{\varphi}^{\alpha}(x) \to 0 \quad \text{as } x \to 0.$$

But φ is continuous on $\mathbb{R}^d$ so

$$\left| \varphi(x - y)^{H - q/\alpha} - \varphi(-y)^{H - q/\alpha} \right|^{\alpha} \to 0 \quad \text{as } x \to 0$$

for almost every $y \in \mathbb{R}^d$. Moreover, arguing as above, as soon as $\tau(x) \leq 1$, for suitable $R > 0$, one can find $C > 0$ such that

$$
\left| \varphi(x-y)^{H-q/\alpha} - \varphi(-y)^{H-q/\alpha} \right|^\alpha
$$
$$
\leq C \left(\tau(y)^{\alpha H - q} \mathbf{1}_{\tau(y) \leq K(R+1)}(y) + \tau(y)^{\alpha(H-\beta)-q} \mathbf{1}_{\tau(y) \geq R}(y) \right),
$$

where $\mathbf{1}_B(y)$ denotes the indicator function of a set B. Then (3.3) holds using dominated convergence.

This concludes the proof. $\qquad \square$

Corollary 3.2. *Under the conditions of Theorem 3.1, the random field $\{X_\varphi(x)\}_{x \in \mathbb{R}^d}$ has the following properties:*

(a) *operator scaling, that is, for any $c > 0$,*

$$
(3.4) \qquad \{X_\varphi(c^E x)\}_{x \in \mathbb{R}^d} \overset{f.d.}{=} \{c^H X_\varphi(x)\}_{x \in \mathbb{R}^d}.
$$

(b) *stationary increments, that is, for any $h \in \mathbb{R}^d$,*

$$
(3.5) \qquad \{X_\varphi(x+h) - X_\varphi(h)\}_{x \in \mathbb{R}^d} \overset{f.d.}{=} \{X_\varphi(x)\}_{x \in \mathbb{R}^d}.
$$

Proof. We will only prove part (a). The proof of part (b) is left to the reader. Fix any $x_1, \ldots, x_m \in \mathbb{R}^d$. Then (3.4) follows if we can show that for any $t_1, \ldots, t_m \in \mathbb{R}$ we have

$$
\sum_{j=1}^m t_j X_\varphi(c^E x_j) \overset{d}{=} c^H \sum_{j=1}^m t_j X_\varphi(x_j).
$$

By a change of variable together with $\varphi(c^E x) = c\varphi(x)$ and the fact that $Z_\alpha(c^E dz) \overset{d}{=} c^{q/\alpha} Z_\alpha(dz)$ we get

$$
\sum_{j=1}^m t_j X_\varphi(c^E x_j) = \int_{\mathbb{R}^d} \sum_{j=1}^m t_j \left(\varphi(c^E x_j - y)^{H-q/\alpha} - \varphi(-y)^{H-q/\alpha} \right) Z_\alpha(dy)
$$
$$
\overset{d}{=} c^{q/\alpha} \int_{\mathbb{R}^d} \sum_{j=1}^m t_j \left(\varphi(c^E(x_j - z))^{H-q/\alpha} - \varphi(-c^E z)^{H-q/\alpha} \right) Z_\alpha(dz)
$$
$$
= c^H \sum_{j=1}^m t_j X_\varphi(x_j)
$$

and the proof is complete. $\qquad \square$

Remark 3.3. Theorem 3.1 and Corollary 3.2 include the following classical isotropic random fields as special cases. Assume $\varphi(x) = \|x\|$ and $E = I$, the identity matrix. Observe that φ is an E-homogeneous, $(1, E)$-admissible function. Then

$$X_\varphi(x) = \int_{\mathbb{R}^d} \left(\|x - y\|^{H-d/\alpha} - \|y\|^{H-d/\alpha} \right) Z_\alpha(dy)$$

Especially, if $\alpha = 2$, then $\{X_\varphi(x)\}_{x \in \mathbb{R}^d}$ is known as the *Lévy fractional Brownian field*. Note that in this case, for any $0 < \alpha \le 2$ equation (3.4) reduces to the well-known self-similarity property $\{X_\varphi(cx)\}_{x \in \mathbb{R}^d} \overset{f.d.}{=} \{c^H X_\varphi(x)\}_{x \in \mathbb{R}^d}$. Moreover our results also include the well known one-dimensional case $d = 1$ of linear fractional stable motions and especially the fractional Brownian motion when $\alpha = 2$.

4. HARMONIZABLE REPRESENTATION

In this section we consider an harmonizable representation of OSSRFs and derive its basic properties. We first give necessary and sufficient conditions such that the integral representation exists and yields a stochastically continuous field. For $0 < \alpha \le 2$, let $W_\alpha(d\xi)$ be a complex isotropic $S\alpha S$ random measure with Lebesgue control measure (see [21] p. 281).

Throughout this section we fix a real $d \times d$ matrix E with $0 < a_1 < \cdots < a_p$ denoting the real parts of the eigenvalues of E. As before, let $q = \text{trace}(E)$.

Theorem 4.1. *Let* $\psi : \mathbb{R}^d \to [0, \infty)$ *be a continuous,* E^t-*homogeneous function such that* $\psi(x) \neq 0$ *for* $x \neq 0$. *Then for any* $0 < \alpha \le 2$ *the random field*

$$(4.1) \qquad X_\psi(x) = \text{Re} \int_{\mathbb{R}^d} \left(e^{i<x,\xi>} - 1 \right) \psi(\xi)^{-H-q/\alpha} W_\alpha(d\xi) \, , x \in \mathbb{R}^d$$

exists and is stochastically continuous if and only if $H \in (0, a_1)$.

Proof. Let us recall that $X_\psi(x)$ exists if and only if

$$\Gamma_\psi^\alpha(x) := \int_{\mathbb{R}^d} \left| e^{i<x,\xi>} - 1 \right|^\alpha \psi(\xi)^{-\alpha H - q} \, d\xi < +\infty.$$

Let us assume that $H \in (0, a_1)$. By integration in *polar coordinates* for E^t given by Proposition 2.3,

$$\Gamma_\psi^\alpha(x) = \int_0^\infty \int_{S_0} \left| e^{i<x, r^{E^t}\theta>} - 1 \right|^\alpha r^{-\alpha H - 1} \psi(\theta)^{-\alpha H - q} \, \sigma(d\theta) \, dr.$$

For $\delta \in (0, H - a_1)$, by considering the cases $r > 1$ and $0 \le r \le 1$ separately and using the same spectral bounds on the growth of $\|r^{E^t}\|$ as in the proof of Lemma 2.1, one can find $C > 0$ such that

$$\left| e^{i<x, r^{E^t}\theta>} - 1 \right|^{\alpha} \le C \left(1 + \|x\|^{\alpha} \right) \min \left(r^{\alpha(a_1 - \delta)}, 1 \right).$$

Moreover, since ψ is continuous with positive values on the sphere S_0, and hence bounded away from zero,

$$\int_{S_0} \psi(\theta)^{-\alpha H - q} \sigma(d\theta) < \infty.$$

This allows us to conclude that $\Gamma_{\psi}^{\alpha}(x)$ is finite for all $x \in \mathbb{R}^d$. Let us show now that X_{ψ} is stochastically continuous. Since X_{ψ} is a $S\alpha S$ field, it is stochastically continuous if and only if, for all $x_0 \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} \left| \left(\left(e^{i<x_0 + x, \xi>} - 1 \right) - \left(e^{i<x_0, \xi>} - 1 \right) \right) \right|^{\alpha} \psi(\xi)^{-\alpha H - q} d\xi \to 0 \quad \text{as } x \to 0$$

that is, equivalently,

(4.2) $$\Gamma_{\psi}^{\alpha}(x) \to 0 \quad \text{as } x \to 0.$$

It is straightforward to see that (4.2) holds for $H \in (0, a_1)$, using dominated convergence and the upper bound computed above.

Conversely, let us assume that X_{ψ} exists and that it is stochastically continuous. Let us remark that in this case $\Gamma_{\psi}^{\alpha}(x)$ exists for all $x \in \mathbb{R}^d$ and satisfies, for all $\lambda > 0$

$$\Gamma_{\psi}^{\alpha}(\lambda^E x) = \lambda^{\alpha H} \Gamma_{\psi}^{\alpha}(x).$$

Let us fix any $x \in \mathbb{R}^d$, with $x \neq 0$ and let us notice that $\Gamma_{\psi}^{\alpha}(x) \neq 0$. Since X_{ψ} is stochastically continuous, by (4.2)

$$\lambda^{\alpha H} \Gamma_{\psi}^{\alpha}(x) \to 0 \quad \text{as } \lambda \to 0,$$

which implies that $H > 0$.

Let us now prove that $H < a_1$.

<u>First case</u>: Assume that a_1 is an eigenvalue of E. Then there exist $\theta_1 \in \mathbb{R}^d$ such that $\|\theta_1\| = 1$ and $E\theta_1 = a_1\theta_1$. Therefore

$$\Gamma_{\psi}^{\alpha}(\theta_1) = \int_0^{\infty} \int_{S_0} \left| e^{i<\theta_1, r^{E^t}\theta>} - 1 \right|^{\alpha} r^{-\alpha H - 1} \psi(\theta)^{-\alpha H - q} \sigma(d\theta) \, dr,$$

with

$$\left| < \theta_1, r^{E^t}\theta > \right| = r^{a_1} | < \theta_1, \theta > | \le C r^{a_1}.$$

Then, for $r \leq \left(\frac{\pi}{C}\right)^{1/a_1}$,

$$\left|e^{i<\theta_1, r^{E^t}\theta>} - 1\right| = 2\left|\sin\left(\frac{<\theta_1, r^{E^t}\theta>}{2}\right)\right| \geq 2r^{a_1}\frac{|<\theta_1, \theta>|}{\pi},$$

and hence

$$\Gamma_\psi^\alpha(\theta_1) \geq \frac{1}{\pi^\alpha} \int_0^{\left(\frac{\pi}{C}\right)^{1/a_1}} \int_{S_0} |<\theta_1, \theta>|^\alpha \, r^{-\alpha(H-a_1)-1} \psi(\theta)^{-\alpha H - q} \, \sigma(d\theta) \, dr.$$

Since ψ is positive on the sphere S_0,

$$\int_{S_0} |<\theta_1, \theta>|^\alpha \, \psi(\theta)^{-\alpha H - q} \, \sigma(d\theta) > 0,$$

and then $\Gamma_\psi^\alpha(\theta_1) < +\infty$ implies that $H < a_1$.

<u>Second case</u>: Assume that a_1 is not an eigenvalue of E. Then there exists $b_1 \in \mathbb{R}$ such that $\lambda_1 = a_1 + ib_1$ and $\overline{\lambda_1}$ are complex eigenvalues of E. One can find $\theta_1, \gamma_1 \in \mathbb{R}^d$, with $\|\theta_1\| = \|\gamma_1\| = 1$ such that

$$r^E\theta_1 = r^{a_1}\left(\cos(b_1 \log r)\theta_1 + \sin(b_1 \log r)\gamma_1\right)$$
$$r^E\gamma_1 = r^{a_1}\left(-\sin(b_1 \log r)\theta_1 + \cos(b_1 \log r)\gamma_1\right).$$

Then it can be shown using the inequality $|e^{i\omega} - 1| \geq |\omega|/\pi$ for $|\omega| < \pi$ that a lower bound of $\Gamma_\psi^\alpha(\theta_1) + \Gamma_\psi^\alpha(\gamma_1)$ is given by

$$\frac{1}{\pi^\alpha} \int_0^{\left(\frac{\pi}{2C}\right)^{1/a_1}} \int_{S_0} \left(|<r^E\theta_1, \theta>|^\alpha + |<r^E\gamma_1, \theta>|^\alpha\right) r^{-\alpha H - 1} \psi(\theta)^{-\alpha H - q} \, \sigma(d\theta) \, dr.$$

Observe that for $a, b \geq 0$ we have $a^\alpha + b^\alpha \geq (a^2 + b^2)^{\alpha/2}$. Therefore

$$|<r^E\theta_1, \theta>|^\alpha + |<r^E\gamma_1, \theta>|^\alpha \geq \left(|<r^E\theta_1, \theta>|^2 + |<r^E\gamma_1, \theta>|^2\right)^{\alpha/2}$$
$$\geq r^{\alpha a_1}\left(|<\theta_1, \theta>|^2 + |<\gamma_1, \theta>|^2\right)^{\alpha/2}.$$

Then we conclude as in the first case that $H < a_1$. The proof is complete. $\qquad\square$

Corollary 4.2. *Under the conditions of Theorem 4.1, the random field $\{X_\psi(x)\}_{x \in \mathbb{R}^d}$ has the following properties:*

(a) *operator scaling, that is, for any $c > 0$,*

(4.3) $$\{X_\psi(c^E x)\}_{x \in \mathbb{R}^d} \overset{f.d.}{=} \{c^H X_\psi(x)\}_{x \in \mathbb{R}^d}.$$

(b) *stationary increments, that is, for any $h \in \mathbb{R}^d$,*

(4.4) $$\{X_\psi(x + h) - X_\psi(h)\}_{x \in \mathbb{R}^d} \overset{f.d.}{=} \{X_\psi(x)\}_{x \in \mathbb{R}^d}$$

Proof. Let us recall that by corollary 6.3.2 of [21], for $f \in L^\alpha(\mathbb{R}^d)$, the characteristic function of the random variable $Y = \mathrm{Re} \int_{\mathbb{R}^d} f(y) W_\alpha(dy)$ is given by

$$(4.5) \qquad \mathbb{E}\left(e^{itY}\right) = \exp\left(-c_0 |t|^\alpha \int_{\mathbb{R}^d} |f(y)|^\alpha dy\right) \quad \text{where} \quad c_0 = \frac{1}{2\pi} \int_0^\pi (\cos\theta)^2 \, d\theta.$$

Hence, for any $x_1, \ldots, x_m \in \mathbb{R}^d$, the finite dimensional characteristic function of $(X_\psi(x_1), \ldots, X_\psi(x_m))$ is given by

$$\mathbb{E}\left(\exp\left(i \sum_{j=1}^m t_j X_\psi(x_j)\right)\right) = \exp\left(-c_0 \int_{\mathbb{R}^d} \left|\sum_{j=1}^m t_j \left(e^{i<x_j,\xi>} - 1\right)\right|^\alpha \psi(\xi)^{-\alpha H - q} d\xi\right),$$

for any $t_1, \ldots, t_m \in \mathbb{R}$. Thus, for any $c > 0$, by a change of variable $\gamma = c^{E^t}\xi$ in the integral of the right side, since ψ is an E^t homogeneous function, we get

$$\mathbb{E}\left(\exp\left(i \sum_{j=1}^m t_j X_\psi(c^E x_j)\right)\right) = \mathbb{E}\left(\exp\left(i \sum_{j=1}^m t_j c^H X_\psi(x_j)\right)\right),$$

which proves (a). Furthermore, for any $h \in \mathbb{R}^d$ and $x \in \mathbb{R}^d$, we have that

$$X_\psi(x+h) - X_\psi(x) = \mathrm{Re} \int_{\mathbb{R}^d} e^{i<h,\xi>}\left(e^{i<x,\xi>} - 1\right) \psi(\xi)^{-H - q/\alpha} W_\alpha(d\xi).$$

Hence

$$\mathbb{E}\left(\exp\left(i \sum_{j=1}^m t_j \left(X_\psi(x_j + h) - X_\psi(x_j)\right)\right)\right)$$

$$= \exp\left(-c_0 \int_{\mathbb{R}^d} \left|\sum_{j=1}^m t_j e^{i<h,\xi>}\left(e^{i<x_j,\xi>} - 1\right)\right|^\alpha \psi(\xi)^{-\alpha H - q} d\xi\right)$$

$$= \mathbb{E}\left(\exp\left(i \sum_{j=1}^m t_j X_\psi(x_j)\right)\right),$$

proving (b). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Remark 4.3. In the Gaussian case, the covariance function of the random field $X_\varphi(x)$ defined by the moving average representation (3.1) can be computed by an argument similar to Proposition 8.1.4 of [21]. Let $\sigma_\theta^2 = \mathbb{E}[(X_\varphi(\theta))^2]$ for any unit vector θ, and define $\tau(x)$ and $\ell(x)$ as before so that $x = \tau(x)^E \ell(x)$. Using Corollary 3.2 (a) it follows that $\mathbb{E}[(X_\varphi(x))^2] = \tau(x)^{2H} \sigma_{\ell(x)}^2$, and then we can use the fact that $2X_\varphi(x)X_\varphi(y) = X_\varphi(x)^2 + X_\varphi(y)^2 - (X_\varphi(x) - X_\varphi(y))^2$ to conclude that

$$(4.6) \qquad \mathbb{E}\left[X_\varphi(x)X_\varphi(y)\right] = \tfrac{1}{2}\left[\tau(x)^{2H}\sigma_{\ell(x)}^2 + \tau(y)^{2H}\sigma_{\ell(y)}^2 - \tau(x-y)^{2H}\sigma_{\ell(x-y)}^2\right].$$

In the isotropic case discussed in Remark 3.3 we have $\tau(x) = \|x\|$ and $\ell(x) = x/\|x\|$, and a change of variables in (3.1) shows that $\sigma_\theta^2 \equiv \sigma^2$ is the same for any unit vector, using the fact that $\varphi(Rx) = \varphi(x)$ for any orthogonal linear transformation R in this case. Then (4.6) reduces to the familiar autocovariance function for a fractional Gaussian random field. A similar argument shows that the autocovariance function of the random field defined by the harmonizable representation (4.1) is given by

$$(4.7) \qquad \mathbb{E}\left[X_\psi(x) X_\psi(y)\right] = \tfrac{1}{2}\left[\tau(x)^{2H}\omega_{\ell(x)}^2 + \tau(y)^{2H}\omega_{\ell(y)}^2 - \tau(x-y)^{2H}\omega_{\ell(x-y)}^2\right]$$

where $\omega_\theta^2 = \mathbb{E}[(X_\psi(\theta))^2]$. For the isotropic case, where (4.1) reduces to the harmonizable representation (1.3) for a fractional Gaussian field, we again note that ω_θ^2 is constant over the unit sphere. Since a mean zero Gaussian random field is determined by its autocovariance function, we recover the well-known fact that the moving average and harmonizable representations of the fractional Gaussian random field differ by at most a constant factor. It does not seem possible to extend this argument to the general case of operator scaling Gaussian random fields, since it would be difficult to compare σ_θ^2 to ω_θ^2 in this case. Hence there remains an interesting open question under which relationship between the functions φ and ψ in the Gaussian case the moving average representation of Theorem 3.1 and the harmonizable representation of Theorem 4.1 are equivalent.

Remark 4.4. Many random fields occurring in applications have Hurst indices that vary with coordinate [7, 8]. Consider a random field satisfying (1.1), and suppose that the matrix E has an eigenvector e with associated real eigenvalue λ. Then it follows from (1.1) that the stochastic process $r \mapsto X(re)$ is self-similar with

$$\{X(c^\lambda re)\}_{r \in \mathbb{R}} \overset{f.d.}{=} \{c^H X(re)\}_{r \in \mathbb{R}} \quad \text{for all } c > 0,$$

so that the Hurst index of this process is H/λ. If E has a basis of eigenvectors with distinct real eigenvalues, then the projections of this random field onto the eigenvector directions yield processes with different Hurst indices in each coordinate. This also shows that the usual methods for estimating the Hurst index, such as rescaled range analysis [15] and dispersional analysis [9], can also be applied to estimate the scaling indices of the operator scaling random field from data, once the proper coordinates are established. Estimating these coordinate directions from data is an interesting open question. In some practical applications, these coordinates are known from

the problem setup. For example, in a groundwater aquifer the coordinates of the hydraulic conductivity field are thought to correspond to the vertical, the direction of horizontal mean flow, and the horizontal direction perpendicular to the mean flow [7]. In fractured rock, the scaling coordinates of the transmissivity field correspond to the main fracture orientations, and are usually not mutually perpendicular [22]. Similarly, in material science, the crack fronts determine the natural coordinates [20]. We caution, however, that estimating the Hurst index in the wrong (non-eigenvalue) coordinates is likely to be misleading, because in those directions the field is not self-similar. Finally, we note that the parameters E, H in (1.1) are not unique. If (1.1) holds, then we also have $\{X(c^{E'}x)\} \stackrel{f.d.}{=} \{cX(x)\}$ where $E' = (1/H)E$, so that the Hurst indices of the random field are the ratio of H and the eigenvalues of E, as already noted. Furthermore, the exponents of an admissible function are not unique, because of possible symmetries, as discussed previously in Remark 2.10. Hence the Hurst index of each component is really an estimate of H/a_i where $0 < a_1 < \cdots < a_p$ is the real spectrum of E, and these indices, as well as the coordinate system in which they pertain, are the same for any choice of H and E.

We have already seen that the OSSRFs, defined by a moving average or a harmonizable representation were stochastically continuous. In the next section we show that in the Gaussian case $\alpha = 2$ one can get Hölder regularity for the sample paths.

5. Gaussian OSSRFs

In this section, we are interested in the smoothness of the sample paths of Gaussian OSSRFs given by Theorem 3.1 or Theorem 4.1, respectively. Moreover we compute the box- and the Hausdorff-dimension of the graph of OSSRFs in these cases. We follow the terminology used in [8]. Using their definition of the Hölder critical exponent of a random process (Definition 5) we state the following definition.

Definition 5.1. Let $\gamma \in (0, 1)$. A random field $\{X(x)\}_{x \in \mathbb{R}^d}$ is said to have Hölder critical exponent γ whenever it satisfies the following two properties:

(a) for any $s \in (0, \gamma)$, the sample paths of X satisfy almost surely a uniform Hölder condition of order s on any compact set, that is for any compact set

$K \subset \mathbb{R}^d$, there exists a positive random variable A such that

$$|X(x) - X(y)| \leq A\|x - y\|^s \quad \text{for all } x, y \in K.$$

(b) for any $s \in (\gamma, 1)$, almost surely the sample paths of X fail to satisfy any uniform Hölder condition of order s.

For a Gaussian random field X a well known result links the Hölder regularity of the sample paths $x \mapsto X(x, \omega)$ to those of the quadratic mean. Let us recall this property when the field also has stationary increments. We refer to [2] Theorem 8.3.2 and Theorem 3.3.2 for a detailed proof.

Proposition 5.2. *Let $\{X(x)\}_{x \in \mathbb{R}^d}$ be a Gaussian random field with stationary incre-ments. Let $\gamma \in (0, 1)$ and assume that*

$$\gamma = \sup \left\{ s > 0; \mathbb{E}\left((X(x) - X(0))^2\right) = o_{\|x\| \to 0}\left(\|x\|^{2s}\right) \right\}.$$

Then, for any $s \in (0, \gamma)$, any continuous version of X satisfies almost surely a uniform Hölder condition of order s on any compact set.
If moreover

$$\gamma = \inf \left\{ s > 0; \|x\|^{2s} = o_{\|x\| \to 0}\left(\mathbb{E}\left((X(x) - X(0))^2\right)\right) \right\},$$

then any continuous version of X admits γ as the Hölder critical exponent.

The previous definition and proposition are given in [8] for random processes ($d = 1$) in order to study regularity properties of a field along straight lines. More precisely, when $\{X(x)\}_{x \in \mathbb{R}^d}$ is a random field, it is also interesting to study the Hölder regularity of the process $\{X(x_0 + tu)\}_{t \in \mathbb{R}}$, for $x_0 \in \mathbb{R}^d$ and u a unit vector. This will provide some additional directional regularity information. For $\{X(x)\}_{x \in \mathbb{R}^d}$ with stationary increments, one only has to consider $\{X(tu)\}_{t \in \mathbb{R}}$ for all directions u. Let us recall Definition 6 of [8].

Definition 5.3. Let $\{X(x)\}_{x \in \mathbb{R}^d}$ with stationary increments and let u be any direction of the unit sphere. If the process $\{X(tu)\}_{t \in \mathbb{R}}$ has Hölder critical exponent $\gamma(u)$ we say that X admits $\gamma(u)$ as directional regularity in direction u.

Let us investigate these properties for the Gaussian OSSRFs given by Theorem 3.1 or Theorem 4.1, respectively. Throughout this section we fix a real $d \times d$ matrix E with $0 < a_1 < \cdots < a_p$ denoting the real parts of the eigenvalues of E. Following

[19], Section 2.1, let $V_1, \ldots, V_p$ be the spectral decomposition of $\mathbb{R}^d$ with respect to E. For $i = 1, \ldots, p$, let us denote

$$W_i = V_1 \oplus \ldots \oplus V_i,$$

and $W_0 = \{0\}$. Observe that $E|_{W_i}$ has $a_1 < \cdots < a_i$ as real parts of the eigenvalues. As before let $q = \mathrm{trace}(E)$.

Theorem 5.4. *Let $\varphi : \mathbb{R}^d \to [0, \infty)$ be an E-homogeneous, (β, E)-admissible function. For $0 < H < \beta$ let X_φ be the moving average Gaussian OSSRF given by Theorem 3.1. Moreover let $\psi : \mathbb{R}^d \to [0, \infty)$ be a continuous E^t-homogeneous function with $\psi(x) > 0$ for all $x \neq 0$. For $0 < H < a_1$ let X_ψ be the harmonizable Gaussian OSSRF given by Theorem 4.1. Then any continuous version of X_φ and X_ψ, respectively, admits H/a_p as Hölder critical exponent. Moreover, for any $i = 1, \ldots, p$, for any direction $u \in W_i \backslash W_{i-1}$, the fields X_φ and X_ψ admit H/a_i as directional regularity in direction u.*

Proof. Let $x \in \mathbb{R}^d$. With a little abuse of notation we write $X_{\varphi/\psi}$ to indicate that we either consider X_φ or X_ψ. Observe that $X_{\varphi/\psi}(0) = 0$ and in order to apply Proposition 5.2 we define

$$\Gamma^2_{\varphi/\psi}(x) = \mathbf{E}\left(X_{\varphi/\psi}(x)^2\right) = \begin{cases} \int_{\mathbb{R}^d} \left|\varphi(x-y)^{H-q/2} - \varphi(-y)^{H-q/2}\right|^2 dy \\ 4 \int_{\mathbb{R}^d} \sin^2\left(\frac{<x,\xi>}{2}\right) \psi(\xi)^{-2H-q} d\xi \end{cases}$$

Using polar coordinates with respect to E, it is straightforward to see that

$$(5.1) \qquad\qquad \Gamma^2_{\varphi/\psi}(x) = \tau(x)^{2H} \Gamma_{\varphi/\psi}(l(x)),$$

where for all $\theta \in S_0$,

$$(5.2) \qquad\qquad 0 < m \leq \Gamma^2_{\varphi/\psi}(\theta) \leq M,$$

since $\Gamma^2_{\varphi/\psi}$ is continuous and positive on the compact set S_0.

For any $i = 1, \ldots, p$ let us fix $u \in W_i \setminus W_{i-1}$. Since the spaces $V_1, \ldots, V_p$ are E-invariant and the real parts of the eigenvalues of $E|_{W_i}$ are $a_1 < \cdots < a_i$ it follows as in the proof of Lemma 2.1 by considering the space W_i instead of $\mathbb{R}^d$, that for any small $\delta > 0$ there exists a constant $C_2 = C_2(u) > 0$ such that $\tau(tu) \leq C_2|t|^{1/a_i - \delta}$ for any $|t| \leq 1$. Furthermore, observe that if we write $u = u_i + \bar{u}_{i-1}$ with $u_i \in V_i$ and $\bar{u}_{i-1} \in W_{i-1}$ we have $u_i \neq 0$. Writing $tu = \tau(tu)^E l(tu)$ and $l(tu) = l_i(tu) + \bar{l}_{i-1}(tu)$ with $l_i(tu) \in V_i$ and $\bar{l}_{i-1}(tu) \in W_{i-1}$, it follows from the E-invariance of the spectral

decomposition that $tu_i = \tau(tu)^E l_i(tu)$ with $l_i(tu) \neq 0$. Since we have $E = E_1 \oplus \cdots \oplus E_p$ where every real part of the eigenvalues of E_i equals a_i we conclude

$$|t| \|u_i\| = \|\tau(tu)^E l_i(tu)\| = \|\tau(tu)^{E_i} l_i(tu)\| \leq \|\tau(tu)^{E_i}\| \|l_i(tu)\| \leq C\tau(tu)^{a_i - \delta}$$

for any $|t| \leq 1$ using the fact that $\|l_i(tu)\| \leq C_3$ for any $|t| \leq 1$ and some $C_3 > 0$. Hence there exists a constant $C_1 = C_1(u) > 0$ such that $\tau(tu) \geq C_1 |t|^{1/a_i + \delta}$ for any $|t| \leq 1$. Therefore we have shown that for all directions $u \in W_i \setminus W_{i-1}$ and any small $\delta > 0$ there exist constants $C_1, C_2 > 0$, such that

$$(5.3) \qquad C_1 |t|^{1/a_i + \delta} \leq \tau(tu) \leq C_2 |t|^{1/a_i - \delta} \quad \text{for all } |t| \leq 1.$$

In view of (5.1), (5.2) and (5.3) we therefore get that for any direction $u \in W_i \setminus W_{i-1}$ and any $\delta > 0$ there exist constants $C_1, C_2 > 0$ such that $C_1 |t|^{2H/a_i + \delta} \leq \Gamma^2_{\varphi/\psi}(tu) \leq C_2 |t|^{2H/a_i - \delta}$ for $|t| \leq 1$, which by Proposition 5.2 shows that $X_{\varphi/\psi}$ admits H/a_i as directional regularity in direction u.

It follows from this that for any $s \in (H/a_p, 1)$ almost surely the sample paths of $X_{\varphi/\psi}$ fail to satisfy any uniform Hölder condition of order s, since H/a_p is the Hölder critical exponent of $X_{\varphi/\psi}$ in any direction of $W_p \setminus W_{p-1}$. Finally, in view of (5.1), (5.2) and Lemma 2.1 we know that for any $\delta > 0$ there exists a constant $C > 0$ such that $\Gamma^2_{\varphi/\psi}(x) \leq C\|x\|^{2H/a_p - \delta}$ for $\|x\| \leq 1$ and hence by Proposition 5.2 it follows that any continuous version of $X_{\varphi/\psi}$ satisfies almost surely a uniform Hölder condition of order $s < H/a_p$ on any compact set. This concludes the proof. $\square$

Having described the Hölder regularity of Gaussian OSSRFs, a natural question that arises is to determine the box- and the Hausdorff-dimensions of their graphs on a compact set. We refer to Falconer [12] for the definitions and properties of box- and the Hausdorff-dimension. Let us fix a compact set $K \subset \mathbb{R}^d$. For a random field X on $\mathbb{R}^d$ we consider $\mathcal{G}(X)(\omega) = \{(x, X(x)(\omega)); x \in K\}$ the graph of a realization of this field over the compact K. We will denote $\dim_{\mathcal{H}} \mathcal{G}(X)$, resp $\dim_{\mathcal{B}} \mathcal{G}(X)$, the Hausdorff-dimension and the box-dimension of $\mathcal{G}(X)$, respectively.

It is a well understood fact that directional regularity implies information about the Hausdorff-dimension of the field in that direction. See e.g. [2], Chapter 8. As an immediate corollary to Theorem 5.4 we get:

Corollary 5.5. *Under the assumptions of Theorem 5.4 we have for all $i = 1, \ldots, p$ and all directions $u \in W_i \setminus W_{i-1}$ that*

$$\dim_{\mathcal{H}}\{(t, X_{\varphi/\psi}(tu)) : t \in [0,1]\} = 2 - H/a_i \quad a.s.$$

Proof. The result is a direct consequence of Theorem 5.4 and the Corollary on page 204 of [2], using the fact that $t \mapsto X_{\varphi/\psi}(tu)$ is a $\beta = H/a_i$-index process and that $2 - \beta \leq 1/\beta$ for $0 < \beta < 1$. $\qquad\square$

Our next result investigates the global box- and Hausdorff-dimension of Gaussian OSSRFs.

Theorem 5.6. *Under the assumptions of Theorem 5.4, for any continuous version of X_φ and X_ψ, almost surely*

$$\dim_{\mathcal{H}} \mathcal{G}(X_{\varphi/\psi}) = \dim_{\mathcal{B}} \mathcal{G}(X_{\varphi/\psi}) = d + 1 - H/a_p.$$

Proof. Let us choose a continuous version of $X_{\varphi/\psi}$. From Theorem 5.4, for any $s < H/a_p$, the sample paths of $X_{\varphi/\psi}$ satisfy almost surely a uniform Hölder condition of order s on K. Thus by a d-dimensional version of Corollary 11.2 of [12], we have

$$\dim_{\mathcal{H}}\mathcal{G}(X_{\varphi/\psi}) \leq \overline{\dim_{\mathcal{B}}}\mathcal{G}(X_{\varphi/\psi}) \leq d + 1 - s, \text{ a.s.}$$

where $\overline{\dim_{\mathcal{B}}}$ denotes the upper box-dimension. Therefore

$$\dim_{\mathcal{H}}\mathcal{G}(X_{\varphi/\psi}) \leq \overline{\dim_{\mathcal{B}}}\mathcal{G}(X_{\varphi/\psi}) \leq d + 1 - H/a_p, \text{ a.s.}$$

and it remains to show that a.s. $\dim_{\mathcal{H}}\mathcal{G}(X_{\varphi/\psi}) \geq d + 1 - H/a_p$. Since the lower box dimension satisfies $\underline{\dim_{\mathcal{B}}}\mathcal{G}(X_{\varphi/\psi}) \geq \dim_{\mathcal{H}}\mathcal{G}(X_{\varphi/\psi})$ the proof is then complete.
We follow the same kind of ideas developed in [6] and [4]. Let $s > 1$. Following the same argument as in Theorem 16.2 of [12], in view of the Frostman criterion (Theorem 4.13 (a) in [12]), if one proves that the integral I_s

$$I_s = \int_{K \times K} \mathbf{E}\left[\left((X_{\varphi/\psi}(x) - X_{\varphi/\psi}(y))^2 + \|x - y\|^2\right)^{-s/2}\right] dx\, dy,$$

is finite, then almost surely $\dim_{\mathcal{H}}\mathcal{G}(X_{\varphi/\psi}) \geq s$.

As before, let $V_1, \ldots, V_p$ denote the spectral decomposition of $\mathbb{R}^d$ with respect to E and let $W_i = V_1 + \cdots + V_i$. We will choose an inner-product $(\cdot, \cdot)$ on $\mathbb{R}^d$ which makes these spaces mutually orthogonal and use the norm $\|x\| = (x, x)^{1/2}$. Since all norms on $\mathbb{R}^d$ are equivalent, this entails no loss of generality.

Since by assumption $s > 1$, the function $(\xi^2 + 1)^{-s/2}$ is in $L^1(\mathbb{R})$ and its Fourier transform, denoted by f_s, is not only in $L^\infty(\mathbb{R})$ but also in $L^1(\mathbb{R})$. Then we can write, using Fourier-inversion (fundamental lemma in [6])

$$(\xi^2 + 1)^{-s/2} = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi t} f_s(t) dt.$$

It follows that

$$\mathbb{E}\left[\left((X_{\varphi/\psi}(x) - X_{\varphi/\psi}(y))^2 + \|x - y\|^2\right)^{-s/2}\right]$$

$$= \frac{1}{2\pi}\|x - y\|^{-s} \int_{\mathbb{R}} \mathbb{E}\left(e^{it\frac{X_{\varphi/\psi}(x) - X_{\varphi/\psi}(y)}{\|x-y\|}}\right) f_s(t) dt$$

$$= \frac{1}{2\pi}\|x - y\|^{-s} \int_{\mathbb{R}} e^{-\frac{t^2}{2}\frac{\mathbb{E}\left(X_{\varphi/\psi}(x) - X_{\varphi/\psi}(y))^2\right)}{\|x-y\|^2}} f_s(t) dt,$$

since $X_{\varphi/\psi}$ is Gaussian. Then, as $f_s \in L^\infty(\mathbb{R})$, one can find $C > 0$ such that

$$\mathbb{E}\left[\left((X_{\varphi/\psi}(x) - X_{\varphi/\psi}(y))^2 + \|x - y\|^2\right)^{-s/2}\right]$$

$$\leq C\|x - y\|^{1-s}\left(\mathbb{E}\left[\left(X_{\varphi/\psi}(x) - X_{\varphi/\psi}(y)\right)^2\right]\right)^{-1/2}$$

$$\leq Cm^{-1}\|x - y\|^{1-s}\tau(x - y)^{-H},$$

according to (5.1) and (5.2) and using the fact that $X_{\varphi/\psi}$ has stationary increments.

Let us choose $A > 0$ such that $K \subset \{x \in \mathbb{R}^d; \|x\| \leq A/2\}$. Then for some constant $C > 0$

$$I_s \leq C \int_{\|x\| \leq A} \|x\|^{1-s}\tau(x)^{-H} dx,$$

as long as the integral in the right hand side is bounded.

If $p = 1$, by Lemma 2.1, for $\delta > 0$, one can find $C > 0$ such that, for $\|x\| \leq A$,

$$\tau(x)^{-H} \leq C\|x\|^{-H/a_p - \delta},$$

and hence I_s is finite as soon as $s < d + 1 - H/a_p - \delta$.

If $p \geq 2$ let us write $x = x_p + y$ for some $x_p \in V_p$ and $y \in W_{p-1}$ and write $x = \tau(x)^E l(x)$ with $l(x) \in S_0$. Decompose $l(x) = l_p(x) + \theta$ with $l_p(x) \in V_p$ and $\theta \in W_{p-1}$. By the direct sum decomposition we see that $x_p = \tau(x)^E l_p(x)$ and $y = \tau(x)^E \theta$. Moreover, since V_p and W_{p-1} are orthogonal in the chosen inner product it follows that $\|x\| \leq A$ implies both $\|x_p\| \leq A$ and $\|y\| \leq A$ in the associated norm. In view of the proof of Lemma 2.1, restricted to the spaces V_p and W_{p-1}, respectively, it follows that for any

$\delta > 0$ and some constants $C_1, C_2 > 0$, if $\|x\| \leq A$ then

$$\|x_p\| \leq C_1 \tau(x)^{a_p - \delta} \text{ and } \|y\| \leq C_2 \tau(x)^{a_1 - \delta}.$$

Then one can find $c > 0$ such that

$$\tau(x)^H \geq c\|x_p\|^{H/a_p + \delta} \text{ and } \tau(x)^H \geq c\|y\|^{H/a_1 + \delta}$$

and thus

$$\tau(x)^H \geq c/2 \left(\|x_p\|^{H/a_p + \delta} + \|y\|^{H/a_1 + \delta} \right).$$

Hence, for any $\delta > 0$

$$I_s \leq C \int_{\|x_p\| \leq A} \int_{\|y\| \leq A} \left(\|x_p\|^2 + \|y\|^2 \right)^{1/2 - s/2} \left(\|x_p\|^{H/a_p + \delta} + \|y\|^{H/a_1 + \delta} \right)^{-1} dy\, dx_p.$$

Let $k = \dim V_p$ and observe that in the present case $1 \leq k \leq d - 1$. By using polar coordinates for both V_p and W_{p-1}, for some constant $C > 0$ we have $I_s \leq C J_s$ where

$$J_s = \int_0^A \int_0^A (u^2 + v^2)^{1/2 - s/2} \left(u^{H/a_p + \delta} + v^{H/a_1 + \delta} \right)^{-1} u^{k-1} v^{d-1-k}\, du\, dv.$$

The change of variables $u = tv$ yields

$$\begin{aligned}
J_s &= \int_0^A \int_0^{A/v} v^{d-s-H/a_p - \delta}(t^2 + 1)^{1/2 - s/2} \left(t^{H/a_p + \delta} + v^{H/a_1 - H/a_p} \right)^{-1} t^{k-1}\, dt\, dv \\
&\leq \left(\int_0^A v^{d-s-H/a_p - \delta} dv \right) \left(\int_0^{+\infty} (t^2 + 1)^{1/2 - s/2} t^{-H/a_p - \delta + k - 1} dt \right).
\end{aligned}$$

Since $\frac{H}{a_p} < 1 \leq k$, the second term is bounded as soon as $s > k + 1 - H/a_p - \delta$, whereas the first one is finite whenever $s < d + 1 - H/a_p - \delta$. Thus, for all $\delta > 0$ small enough, it follows that almost surely $\dim_{\mathcal{H}} \mathcal{G}(X_{\varphi/\psi}) \geq d + 1 - H/a_p - \delta$ and the proof is complete. $\qquad \square$

Remark 5.7. As pointed out in the introduction, the fractional Brownian sheet $\{B_H(x)\}_{x \in \mathbb{R}^d}$ is operator scaling with $E = \operatorname{diag}(\alpha_1, \ldots, \alpha_d)$ where $\alpha_i = 1/H_i$, in fact $\{B_H(c^E x)\}_{x \in \mathbb{R}^d} \overset{f.d.}{=} \{cB_H(x)\}_{x \in \mathbb{R}^d}$, but does not have stationary increments. By Theorem 1.1 of [3] we know that $\dim_{\mathcal{H}} \mathcal{G}(B_H) = \dim_{\mathcal{B}} \mathcal{G}(B_H) = d + 1 - \min(\alpha_1^{-1}, \ldots, \alpha_d^{-1})$. Now let $X_{\varphi/\psi}$ be our Gaussian OSSRFs considered above. It follows from operator scaling, that

$$\{X_{\varphi/\psi}(c^{\frac{1}{H}E} x)\}_{x \in \mathbb{R}^d} \overset{f.d.}{=} \{cX_{\varphi/\psi}(x)\}_{x \in \mathbb{R}^d}$$

too, and that $\bar{\alpha}_i = a_i/H$, $i = 1, \ldots, p$ are the real parts of the eigenvalues of $(1/H)E$. Hence Theorem 5.6 can be reformulated as

$$\dim_{\mathcal{H}} \mathcal{G}(X_{\varphi/\psi}) = \dim_{\mathcal{B}} \mathcal{G}(X_{\varphi/\psi}) = d + 1 - \min(\bar{\alpha}_1^{-1}, \ldots, \bar{\alpha}_p^{-1})$$

in complete similarity to the result in [3]. Hence we have constructed operator scaling Gaussian random fields with the same box- and Hausdorff-dimension as the fractional Brownian sheet, but our fields have additionally stationary increments.

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HERMINE BIERMÉ, MAP5 UNIVERSITÉ RENÉ DESCARTES, 45 RUE DES SAINTS PÈRES, 75270 PARIS CEDEX 06, FRANCE

E-mail address: `hermine.bierme@math-info.univ-paris5.fr`

MARK M. MEERSCHAERT, DEPARTMENT OF MATHEMATICS & STATISTICS, UNIVERSITY OF OTAGO, NEW ZEALAND

E-mail address: `mcubed@maths.otago.ac.nz`

URL: `http://www.maths.otago.ac.nz/ mcubed/`

HANS-PETER SCHEFFLER, FACHBEREICH MATHEMATIK, UNIVERSITÄT SIEGEN, 57068 SIEGEN, GERMANY

E-mail address: `peter.scheffler@mac.com`